AUTONETICS
A DIVISION OF NORTH AMERICAN AVIATION, INC.
INDUSTRIAL PRODUCTS
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RECOMP TECHNICAL BULLETIN NO. 22.1

SUBJECT: GENERATION OF RANDOM NUMBERS ON THE RECOMP DIGITAL COMPUTER

PURPOSE: This bulletin describes techniques for generating sequences of pseudo-random numbers with the RECOMP digital computer and discusses methods for obtaining various statistical and spectral properties in such sequences.

EFFECTIVE DATE: 13 September 1961

REFERENCES:
(a) "Quarterly of Applied Mathematics" Vol XVI, 1958.

(b) "Symposium on Monte Carlo Methods", edited by H. A. Meyer, John Wiley and Sons, 1956; in particular, pp 15-28, Olga Taussky and John Todd, "Generation and Testing of Pseudo-Random Numbers".

(c) Martin Greenberger "Notes on a New Pseudo-Random Number Generator" J. Assoc. Comp. Mach. Vol. 8, No. 2, April 1961.

(d) J. N. Franklin, "Note on the Equidistribution of Pseudo-Random Numbers, Quarterly of Applied Mathematics, Vol. XVI No. 2, July 1958.

INFORMATION TO: All Concerned

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System Integration
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In system analysis work there arises from time to time a need for sequences
of random numbers, to simulate, e.g., the effects of noise and errors of
a random nature on system performance; and, of course, such sequences are
essential to the use of Monte Carlo techniques. This memorandum will discuss
some techniques for generating sequences of numbers which are suitable for
these puproses, and will describe some experiments that have been conducted
to study the nature of such sequences. Although the following discussion will
be limited to application to Autonetics' RECOMP digital computer, the basic
ideas are certainly applicable to digital computation in general.

A conventional digital computer such as RECOMP is, of course, incapable of
performing a truly random process; all of its operations are deterministic.
However, there exists a rather convenient method of generating a sequence of
numbers which, from the standpoint of the user are (through his ignorance,
if you will) unpredictable and in this sense are pseudo-random.   By a pseudo-
random sequence we mean a previously determined sequence which is used to
simulate a random sequence; however, in the discussion that follows the pre-
fix "pseudo" will be omitted when we refer to such sequences.

The method for generating this sequence is as follows: Let N be an integer
greater than one and let x_0 be a fraction

$$0 < x_0 < 1$$

Define

$$x_{k+1} = \text{fractional part of } \left[Nx_k \right]$$

Then the sequence $\{x_k\}$ is uniformly distributed between zero and one.

That is, the probability density function

$$p(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

To generate this random sequence on RECOMP we store the fraction x at
a binary scale of zero and the integer N at a binary scale of 39. Then
if we multiply N by x we have the fractional part of the product, which
is the new x, in the R register. The sequence of commands is

```
CLA     x
MPY     N
XAR            (exchange A & R registers)
STO     x
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The simplicity of this method is apparent. In practice it is recommended that $X_0 = 2^{-39}$ and that an odd power of 3 or 5 be selected for N. Different choices will of course provide different sequences. It is best to select the largest odd power of 3 or 5 that can be contained in 39 bits. For further discussion on this point, as well as the mathematical nature of the pseudo-random sequence, see references (b) and (c). A convenient method of obtaining different sequences is to employ two generators with different odd powers of, say, 5 for N. A member of the first sequence is selected, with an element of chance, by throwing a sense switch, to be used as the starting number for the second sequence which provides the random numbers for the problem. A brief table of odd powers of 3 and 5 in command format is given in Appendix B.

In a practical application we are concerned with two characteristics of the random sequence. One of these is the distribution of the random numbers. As asserted above the members of the random sequence are uniformly distributed between zero and one, and a proof of this fact may be found in reference (d). Figure 1 shows the actual distribution of a sample of 1024 numbers generated in this manner.

A second characteristic of interest is the sequencing of the numbers, or more precisely, if the sequence is thought of as a random time series, the power spectral density of the series. In this regard experimental evidence indicates that the spectral density is "white" or uniform over all frequencies. (Of course, as in all digital computer work, the spectral content is band limited in a real time sense, by the sampling frequency, or the rate at which the numbers are generated.) Another way of considering this characteristic which does not depend on any concept of time is to state that the members of the sequence are statistically independent of one another. To demonstrate this fact an "autocorrelation" function was computed.

$$R_x(\ ) = \frac{1}{N} \sum_{K=1}^{N} x_k\, x_{k+} \qquad = 0\ 1\ 2\ \ldots$$

Figure 2 shows an actual example of this computation on a sample of 1024 numbers. (The numbers were shifted first by subtracting one-half to lie between plus and minus one-half.) From this figure it is apparent that for one or more shifts the numbers are uncorrelated.

An important consequence of the independence of the members of the sequence is that it permits the use of a single random number generator to provide numbers for several applications in the same problem.
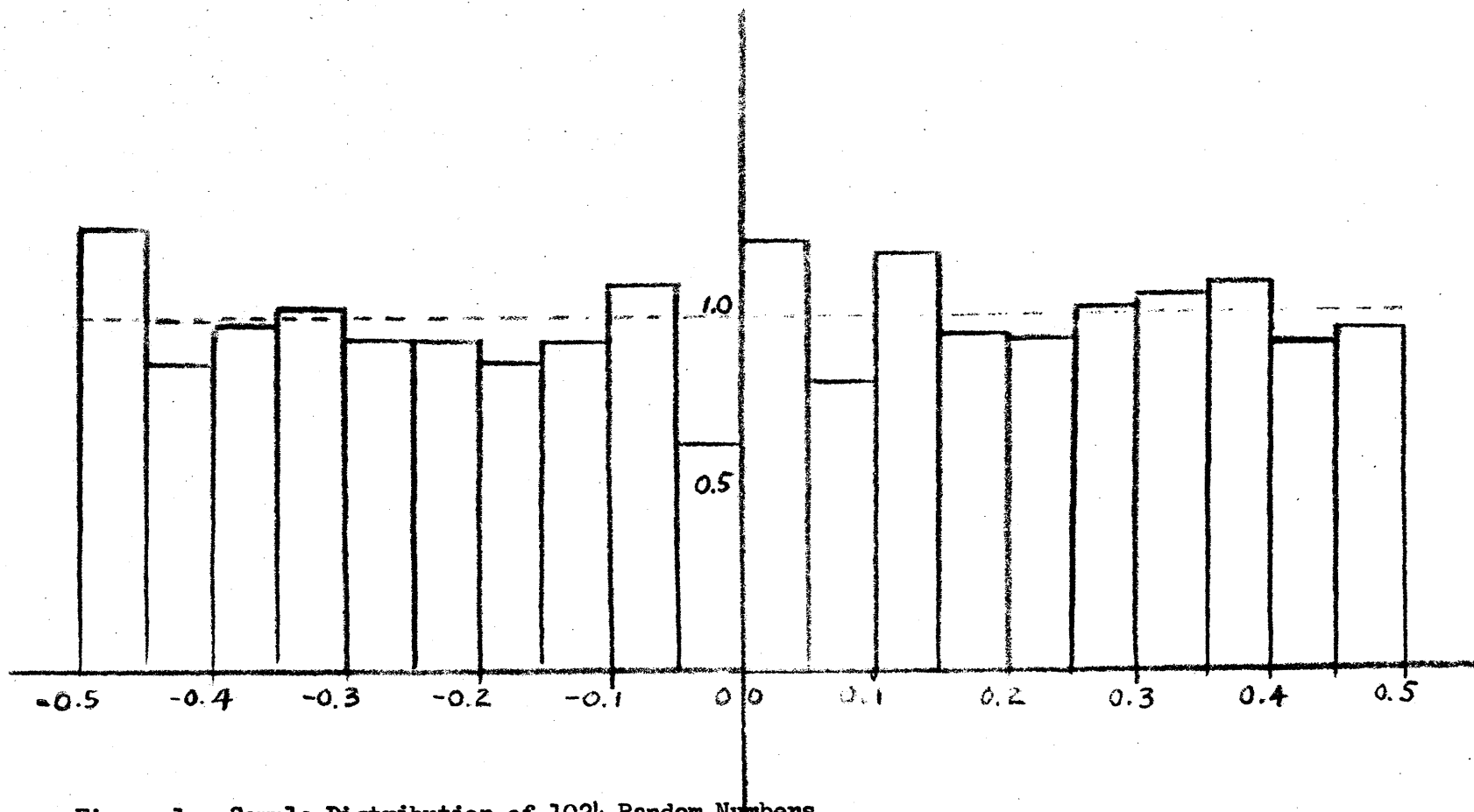
Figure 1. Sample Distribution of 1024 Random Numbers.

$R_x(\tau)$

$R_x(0) = \sigma_x^2 = .0848$

0.08

0.06

0.04

0.02

0.00

$$R_x(\tau) = \frac{1}{N} \sum_{K=1}^{N} X_K X_{K+\tau} \qquad \tau = 0, 1, \ldots, 31$$

$$N = 1024$$

with X uniformly distributed between $-\frac{1}{2}$ and $+\frac{1}{2}$

5   10   15   20   25   30 τ

τ
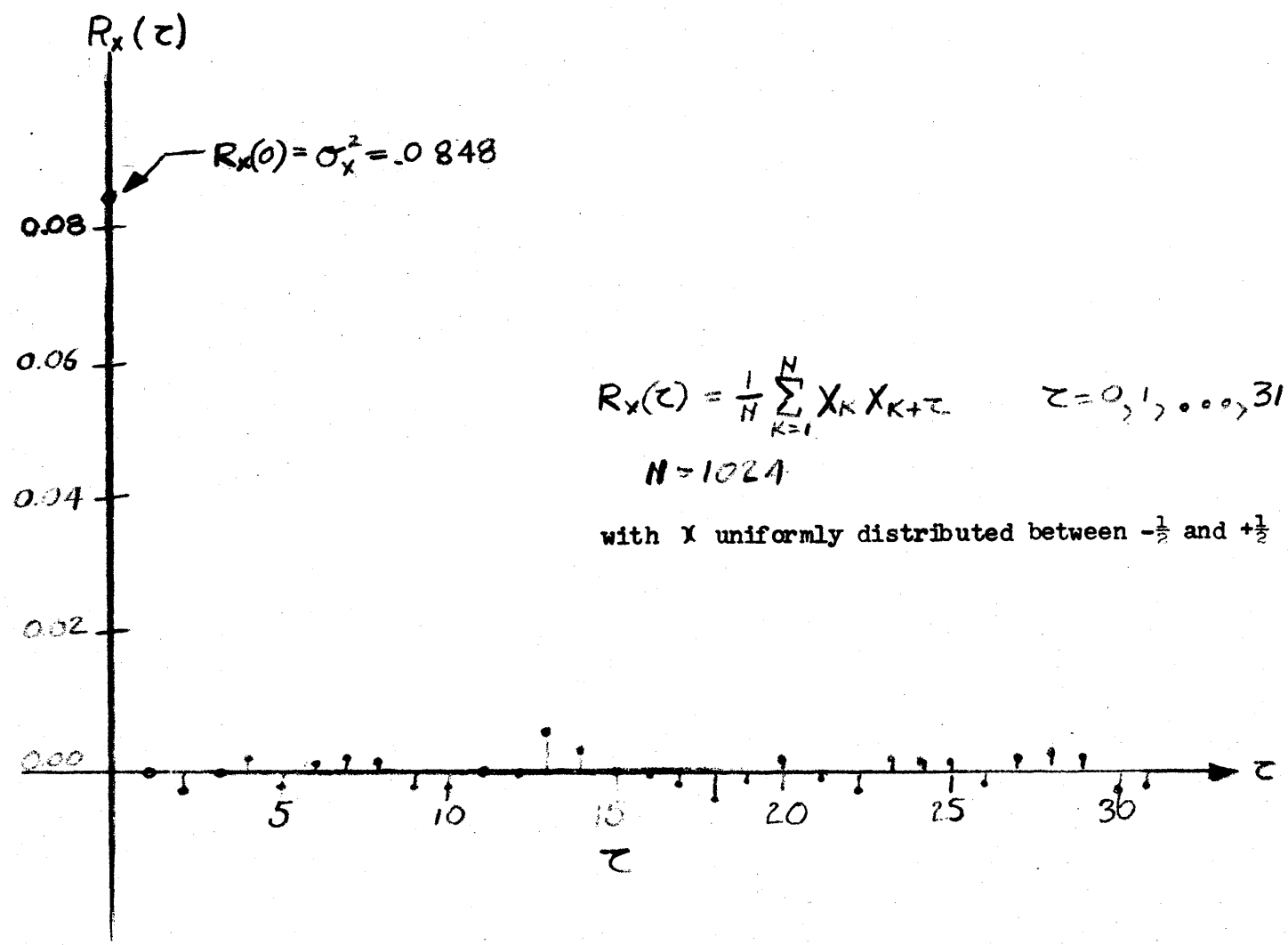
Figure 2.   Autocorrelation Function of 1024 Random Numbers.

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Thus, we have available a simple method of generating a random sequence whose members are uniformly distributed between zero and one and are independent of one another or "white". Other rectangular distributions may be easily obtained by multiplying by a scale factor and adding a bias. For example, suppose it is desired to select at random an integer between 1 and 52. We simply generate a random number, multiply by 52, add 1, and take the integral part of the answer as the desired number.

However, distributions other than rectangular are often required. For example, failure rate is often characterized by an exponential distribution, system errors are frequently considered to have gaussian distributions; other random events may have a Poisson distribution. How may other distributions be obtained from a rectangular distribution?

If x is uniformly distributed between zero and one and if the probability density function p (t) has the properties

$$p\,(t) \geq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} p\,(t)\,dt = 1$$

then the random variable z defined by the relation

$$x = \int_{-\infty}^{z} p(t)\,dt \tag{1}$$

has the probability distribution function

$$P\,(z) = \int_{-\infty}^{z} p\,(t)\,dt$$

and hence the probability density function p (z). For Probability $\left\{ z \leq z_0 \right\}$

$$= \text{Probability} \left\{ x \leq \int_{-\infty}^{z_0} p\,(t)\,dt \right\}$$

$$= \int_{-\infty}^{z_0} p\,(t)\,dt$$

since x is uniformly distributed between zero and one and hence Probability

$$\left\{ x \leq a \right\} = a$$

For example, suppose an exponential distribution is required, i.e.,

$$p(z) = \begin{cases} ae^{-az} & 0 \leq z \\ 0 & z < 0 \end{cases}$$

- -

From equation (1) it is therefore required that the uniform random variable

$$x = \begin{cases} \int_0^z ae^{-at}\, dt & 0 \leq z \\ 0 & z < 0 \end{cases}$$

For $z \geq 0$

$$x = 1 - e^{-az}$$

or

$$z = \frac{1}{-a} \log(1-x)$$

or since x and $(1 - x)$ have the same distribution, we let

$$z = \frac{1}{-a} \log x$$

and the random variable z will have the probability density $p(z)$ as desired. In other words, we generate a random number, compute its natural logarithm, and divide by minus a. The result will have the probability density function $p(z)$.

As a second example suppose it is desired to select a point at random from the unit circle under the assumption that the points are uniformly distributed over the circle. One possible solution would be to generate two random numbers, say x and y, scaled and biased to lie between plus and minus one, rejecting the pair if the sum of their squares exceeded one. The pairs (x, y) that were accepted would, of course, have a uniform distribution over the unit circle.

An alternate method would use polar coordinates r and θ. From symmetry it is clear that any angle is equally likely so the probability density of θ is given by

$$p_\theta(\theta) = \begin{cases} \dfrac{1}{2\pi} & 0 < \theta < 2\pi \\ 0 & \text{otherwise} \end{cases}$$

Therefore, to generate the random variable θ we simply generate a random number and multiply it by 2π.

To find the distribution of r we note that the probability of a point lying in an incremental area is

$$\frac{dx\ dy}{\pi} = \frac{r\ dr\ d\theta}{\pi}$$

$$= (2r\ dr)\ \frac{d\theta}{2\pi}$$

$$= p_r(r)dr\ p_\theta(\theta)\ d\theta$$

Since we have already determined $p_\theta(\theta)$ it follows that

$$p_r(r) = \begin{cases} 2r & 0 < r \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

From equation (1) we require that the uniformly distributed number

$$x = \int_0^r 2t\ dt = r^2$$

or

$$r = x^{\frac{1}{2}}$$

To summarize the procedure then we generate two random numbers. One of these is multiplied by 2π and designated θ . The square root of the other is extracted and the result designated r. The resulting points (r,θ) are uniformly distributed over the unit circle. The size of the circle can be easily scaled by multiplying r by the radius of the desired circle.

In principle any desired distribution may be obtained by the method discussed in the preceding paragraphs. However, there may be computational difficulties for some density functions. Unfortunately, this is the case for the most important gaussian distribution,

$$p(z) = \frac{1}{\sqrt{2\pi}}\ e^{-z^2/2}$$

According to equation (1) it would be necessary to invert the equation

$$x = \int_{-\infty}^z \frac{e^{-t^2/2}}{\sqrt{2\pi}}\ dt$$

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If the function on the right is designated $\underline{\Phi}(z)$ we see that it is required to find

$$z = \underline{\Phi}^{-1}(x)$$

and of course this equation may be solved by numerical methods; but, it would appear that considerable computation would be required, and for this reason it has not been attempted. As an alternate approach we again resort to polar coordinates. If u and v are independent random numbers with gaussian distributions then the random variable

$$r = \sqrt{u^2 + v^2}$$

has the so-called Rayleigh distribution

$$p(r) = \begin{cases} re^{-r^2/2} & r \geq 0 \\ \\ 0 & r < 0 \end{cases}$$

and the random variable

$$\Theta = \tan^{-1} \frac{v}{u}$$

is uniformly distributed between zero and 2π. To obtain a random number with the Rayleigh distribution, as before, we let the uniformly distributed number

$$x = \int_0^r te^{-t^2/2}\, dt$$

$$= 1 - e^{-r^2/2}$$

or

$$r = \left[2 \log \left(\frac{1}{1-x} \right) \right]^{\frac{1}{2}}$$

or since x and $(1-x)$ have the same distribution we simply let

- -

$$r = \left[2 \log \frac{1}{x} \right]^{\frac{1}{2}}$$

Similarly we let

$$\Theta = 2\pi x$$

where of course x is another uniformly distributed number. Then we simply make the inverse transformation

$$u = r \cos \Theta$$

$$v = r \sin \Theta$$

and u and v will be independent random numbers with gaussian distributions. It will be noted that this method requires the computation of a logarithm and sine and cosine for which subroutines are normally available.

The above method provides a pair of independent gaussian random numbers at the expense of computing a logarithm and sine-cosine. Since the latter computations are somewhat time consuming, and usually time is at a premium in analyses of a probabilistic nature, this method has limited practical application. The following approach provides a technique for generating numbers whose distribution is approximately gaussian with relatively little computation required.

From the central limit theorem of statistics it is known that the distribution of the average of N samples from any distribution approaches the gaussian distribution as N becomes large. If the original distribution is rectangular, the convergence is remarkably fast, in fact, N = 3 or 4 gives a distribution which is a very good approximation to the gaussian.

Let

$$p_1(x_i) = \begin{cases} 1 & 0 \le x_i < 1 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

for i = 1, 2, . . . , N

Then the random variable

$$z = \sum_{i=1}^{N} x_i$$

has the probability density function p_n (z) which may be obtained by convolving p_1 with p_{N-1} or

$$p_n (z) = \int_{-\infty}^{\infty} p_1(z-x) \, p_{n-1} (x) \, dx \quad N = 2, 3, \ldots.$$

For convenience we let

$$u = \frac{z - \mu_n}{\sigma_n}$$

where

$$\mu_n = \int_{-\infty}^{\infty} t p_n(t)dt$$

$$\sigma_n^2 = \int_{-\infty}^{\infty} (t - \mu_n)^2 \, p_n(t)dt$$

so that u_n has a zero mean and unit variance.

For N = 2 the distribution is triangular shaped and of limited interest. For N = 3 evaluation of the convolution integral gives

$$p_3(u) = \begin{cases} \dfrac{3 - u^2}{8} & 0 \leq |u| \leq 3 \\[2mm] \dfrac{(3 - |u|)^2}{16} & 1 \leq |u| \leq 3 \\[2mm] 0 & 3 \leq |u| \end{cases} \tag{3}$$

where $u = 2 (x_1 + x_2 + x_3) - 3$ $\tag{4}$

with x_i, i = 1, 2, 3, distributed according to (2). A plot of p_3 (u) is given in Figure 3. For comparison, a plot of the gaussian distribution

$$\psi (u) = \frac{1}{\sqrt{2\pi}} \, e^{-u^2/2}$$

is given on the same figure. From this figure it is seen that the distribution

$$\Psi(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$$

$p_3(u)$ = DISTRIBUTION OF $u = 2(X_1 + X_2 + X_3) - 3$

WHERE $p(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{OTHERWISE} \end{cases}$
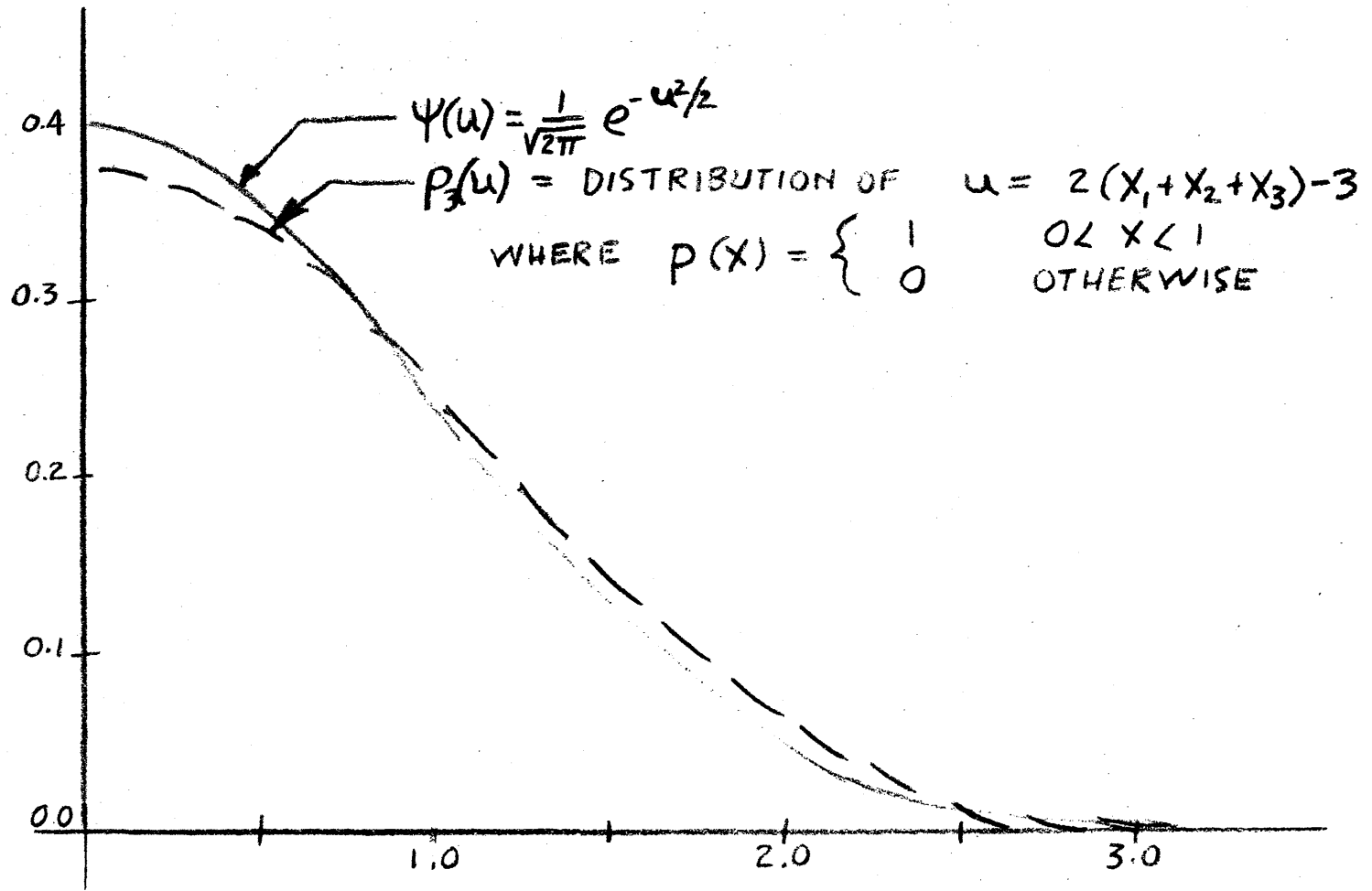
Figure 3.   Comparison of Distribution p_3 (u) with Gaussian.

of u is a very good approximation to the gaussian. In fact, it would be
difficult to distinguish between samples from the two distributions unless
an extremely large sample were taken. Figure 4 shows the actual distri-
bution of 5000 numbers generated according to equation (4). In this
figure the area of the rectangles equals the fraction of the sample falling
in the corresponding interval. The gaussian is also plotted on this figure
for comparison.

This technique lends itself readily to digital computation. A subroutine for
generating "approximate gaussian" random numbers in this fashion is given in
Appendix A. As mentioned above, these numbers have the distribution given
by (3) with zero mean and unit variance. To simulate the gaussian random
variable z with mean μ_z and variance σ_z, let

$$z = \sigma_z u + \mu_z$$

It should be noted that members of the resulting gaussian random sequence
are also independent of one another or "white".

An even better approximation may be obtained by combining four of the uni-
formly distributed numbers. For N equal 4, evaluation of the convolution
integral yields

$$p_4(u) = \begin{cases} \dfrac{4\sqrt{3} - 2\sqrt{3}\, u^2 + |u|^3}{18} & 0 \le |u| \le \sqrt{3} \\[3mm] \dfrac{(2\sqrt{3} - |u|)^3}{54} & \sqrt{3} \le |u| \le 2\sqrt{3} \\[3mm] 0 & 2\sqrt{3} \le |u| \end{cases}$$

where $u = \sqrt{3}\left[\sum_{i=1}^{4} x_i - 2\right]$

with x_i, i = 1, 2, 3, 4, distributed according to (2). As before the distri-
bution has been scaled to have a zero mean and unit variance. Figure 5 shows
a plot of $p_4(u)$ with the gaussian for comparison.

Appendix C provides a tabulation of the density functions $p_3(u)$ and $p_4(u)$
together with their respective cumulative distributions.
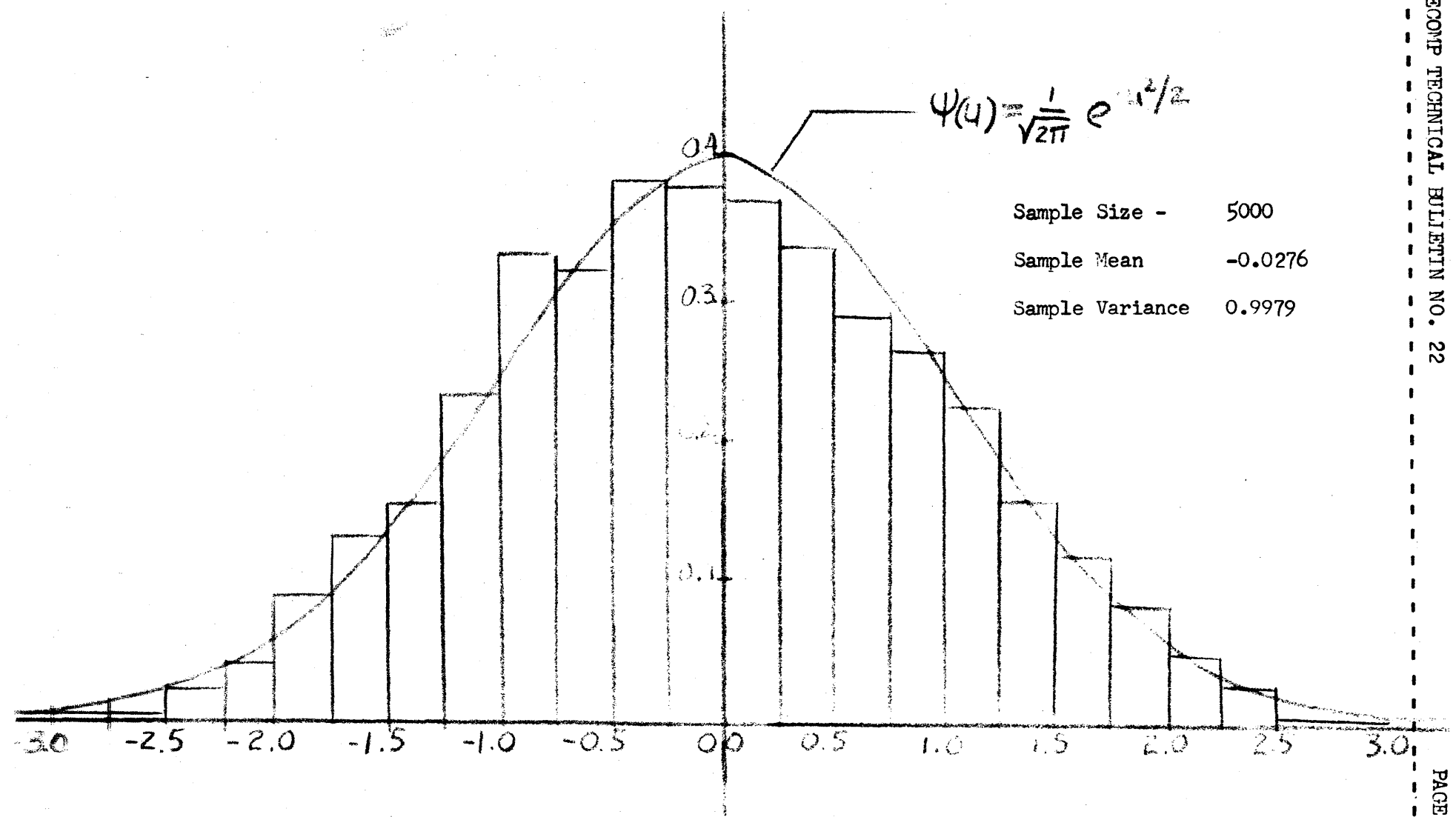
Figure 4. p_3 (u) Sample Distribution.

$$\psi(u) = \frac{1}{\sqrt{2\pi}} \, e^{-u^2/2}$$

Sample Size –	5000
Sample Mean	-0.0276
Sample Variance	0.9979
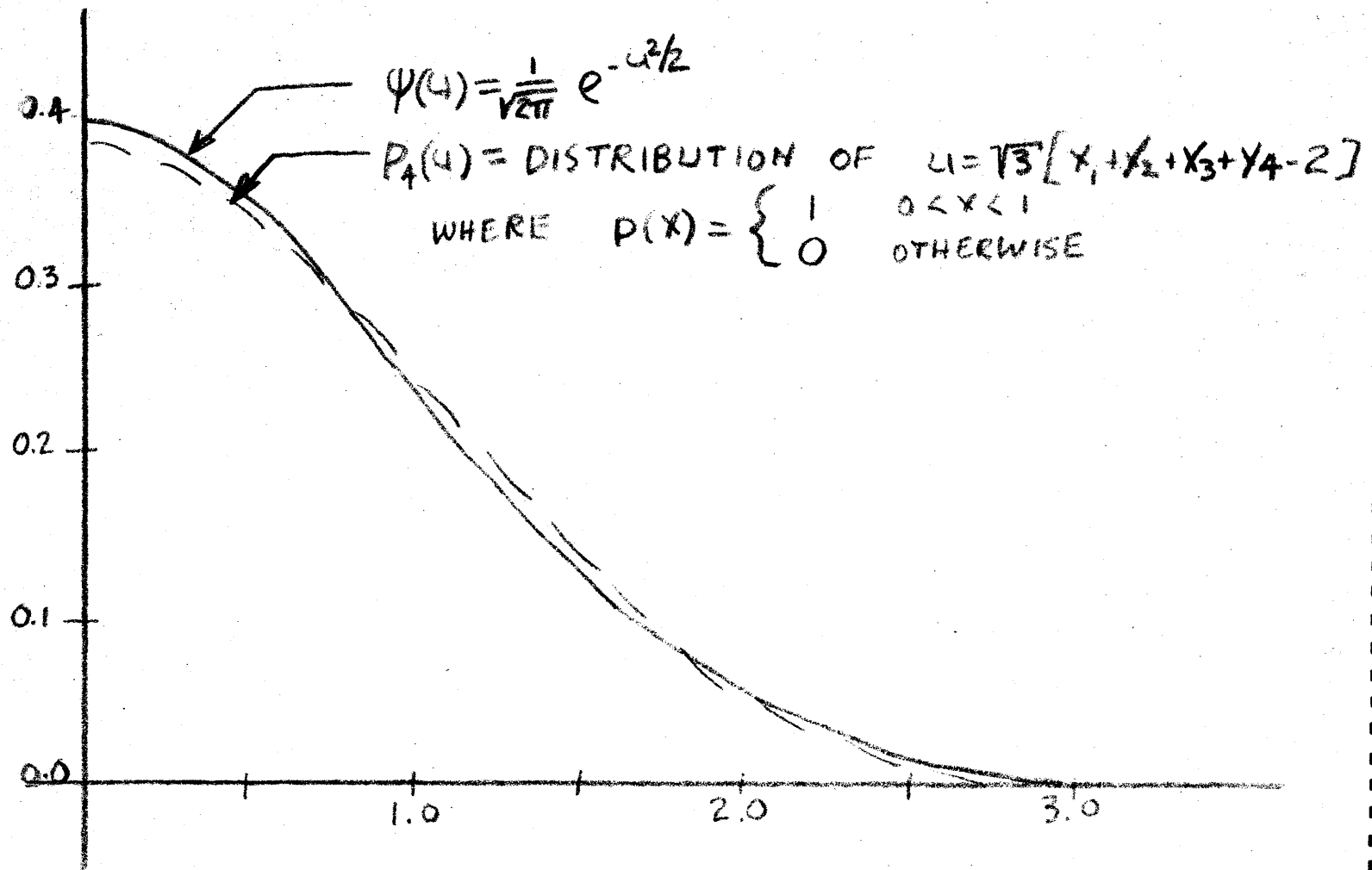
Figure 5. Comparison of Distribution p_4 (u) with Gaussian.

$$\psi(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$$

$p_4(u) = $ DISTRIBUTION OF $u = \sqrt{3} \left[X_1 + X_2 + X_3 + Y_4 - 2 \right]$

WHERE $p(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{OTHERWISE} \end{cases}$

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Another technique that is useful in providing a distribution that, although
not gaussian, favors small numbers over large and thus may be adequate for
some purposes, is to simply multiply two of the uniformly distributed numbers.
If x_1, x_2 are distributed according to

$$p(x_i) = \begin{cases} \tfrac{1}{2} & -1 \leq x_i \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$i = 1, 2$$

then the random variable $z = x_1 x_2$

has the distribution

$$p(z) = \begin{cases} \tfrac{1}{2} \log \dfrac{1}{|z|} & -1 \leq z \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Also the random variable $y = x_1 |x_1|$ has the distribution

$$p(y) = \begin{cases} \dfrac{1}{4 |y|^{\tfrac{1}{2}}} & -1 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Random variables such as these have the advantage of being computed rather
easily.

As a final subject, we will briefly consider the case where a random sequence
is required with a spectral density other than white. For example, it might
be required to constrain the sequence so that it does not change value too
rapidly or in other words, suppress the high frequency components in the sequence.
To achieve this end it is necessary to run the "white" sequence through a low-
pass filter.

If the input $x(t)$ to a filter with transfer function $H(jw)$ is white, i.e.,
has power spectral density,

$$S_x(jw) = S_x(0)$$

then the power spectral density of the output $y(t)$ is

$$S_y(jw) = |H(jw)|^2 S_x(0)$$

- -

or in other words, the output spectral content is determined by the filter characteristic. If the input to a linear filter has a gaussian distribution then the output will also be gaussian. It is then only necessary to determine the mean and variance of the output in order to completely characterize the output distribution. Well known techniques are available for the design of digital filters and thus it is possible to generate gaussian random numbers with a desired spectral content.

To demonstrate this technique a simple first order lag filter was programmed and fed with a white gaussian input. The distribution and autocorrelation function of the output were then computed. Such a filter has the characteristic

$$H(jw) = \frac{a}{a + jw}$$

and impulse response

$$h(t) = ae^{-at}$$

where   a   is the so-called corner frequency. The output spectral density is therefore

$$S_y(jw) = \frac{a^2}{a^2 + w^2} \quad S_x(0) \tag{7}$$

The autocorrelation function of the output (the inverse transform of $S_y(jw)$

has the form

$$R_y(\tau) = \sigma_y^2 \, e^{-a|\tau|} \tag{8}$$

where σ_y is the variance of the output.

To derive the difference equation defining the digital filter, we use the fact that the output is the convolution of the input with the filter impulse response,

$$y(t) = \int_0^t h(t - t') \, x(t') \, dt'$$

$$= \int_0^t ae^{-a(t-t')} \, x(t') \, dt'$$

To evaluate this integral numerically we take an integration step $\Delta t = T$ and assume $x(t)$ is constant over this interval, thus

$$x(t) = x_n \qquad (n-1)T < t < nT \qquad n = 1, 2, \ldots$$

The x_n will be the input white gaussian random sequence, with zero mean and unit variance.

Now

$$y_{n+1} = y((n+1)T) = e^{-a(n+1)T} \int_0^{(n+1)T} ae^{at'} x(t')dt'$$

$$= e^{-at} e^{-anT} \int_{\,}^{nT} ae^{at'} x(t')dt' + e^{-a(n+1)T}\int_{nT}^{(n+1)T} ae^{at'} x_{n+1}dt'$$

or

$$y_{n+1} = e^{-aT} y_n + (1 - e^{-aT}) x_{n+1}$$

Equation (9) is the difference equation that defines the filter. By averaging both sides of (9) we see that the output mean

$$\mu_y = e^{-aT}\mu_y + (1 - e^{-aT})\mu_x$$

or $\mu_y = \mu_x = 0$ since the input has zero mean. The variance of the output may be obtained by squaring both sides and averaging, and noting that the cross term on the right involving $x_{n+1}\, y_n$ averages to zero, since future input x_{n+1} is uncorrelated with present output y_n.

Thus

$$\sigma^2_y = e^{-2aT}\sigma^2_y + (1 - e^{-aT})^2 \sigma^2_x$$

or

$$\sigma^2_y = \frac{1 - e^{-aT}}{1 + e^{-aT}} \sigma^2_x$$

From equations (8) and (10) the autocorrelation function of the output is

$$R_y(\zeta) = \frac{1 - e^{-aT}}{1 + e^{-aT}} \; e^{-\zeta aT}$$

for $\zeta = 0, 1, 2, \ldots$

As an example of this computation we let

$$aT = 0.4$$

This gives

$$\sigma_y^2 = .197$$

Figure 6 shows the distribution of the output (normalized to have unit variance) and Figure 7 shows the output autocorrelation function. One could apply Fourier techniques to determine the output spectral content. However, from the exponential nature of the output autocorrelation function, it is clear that the output has the desired spectral density.

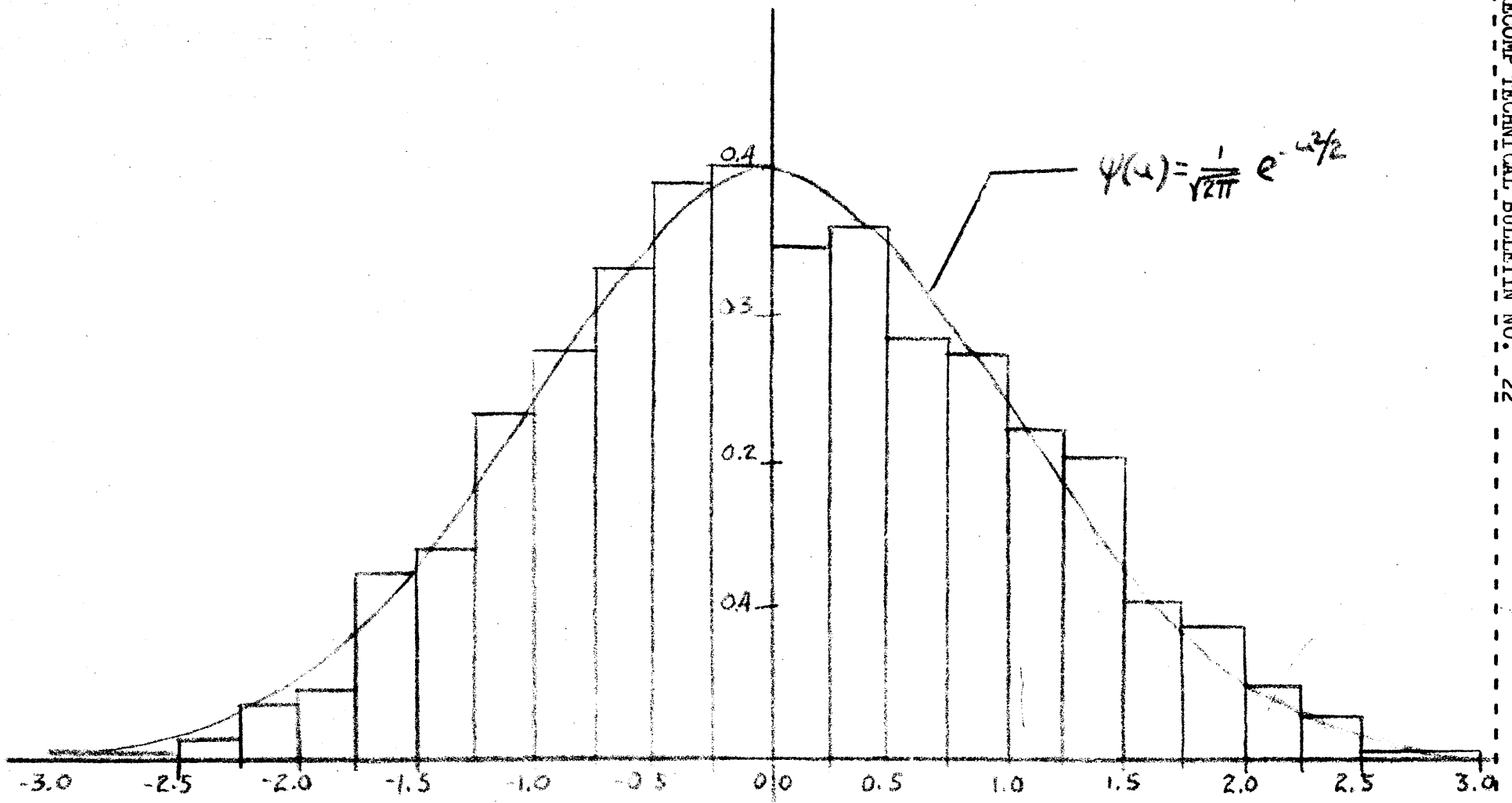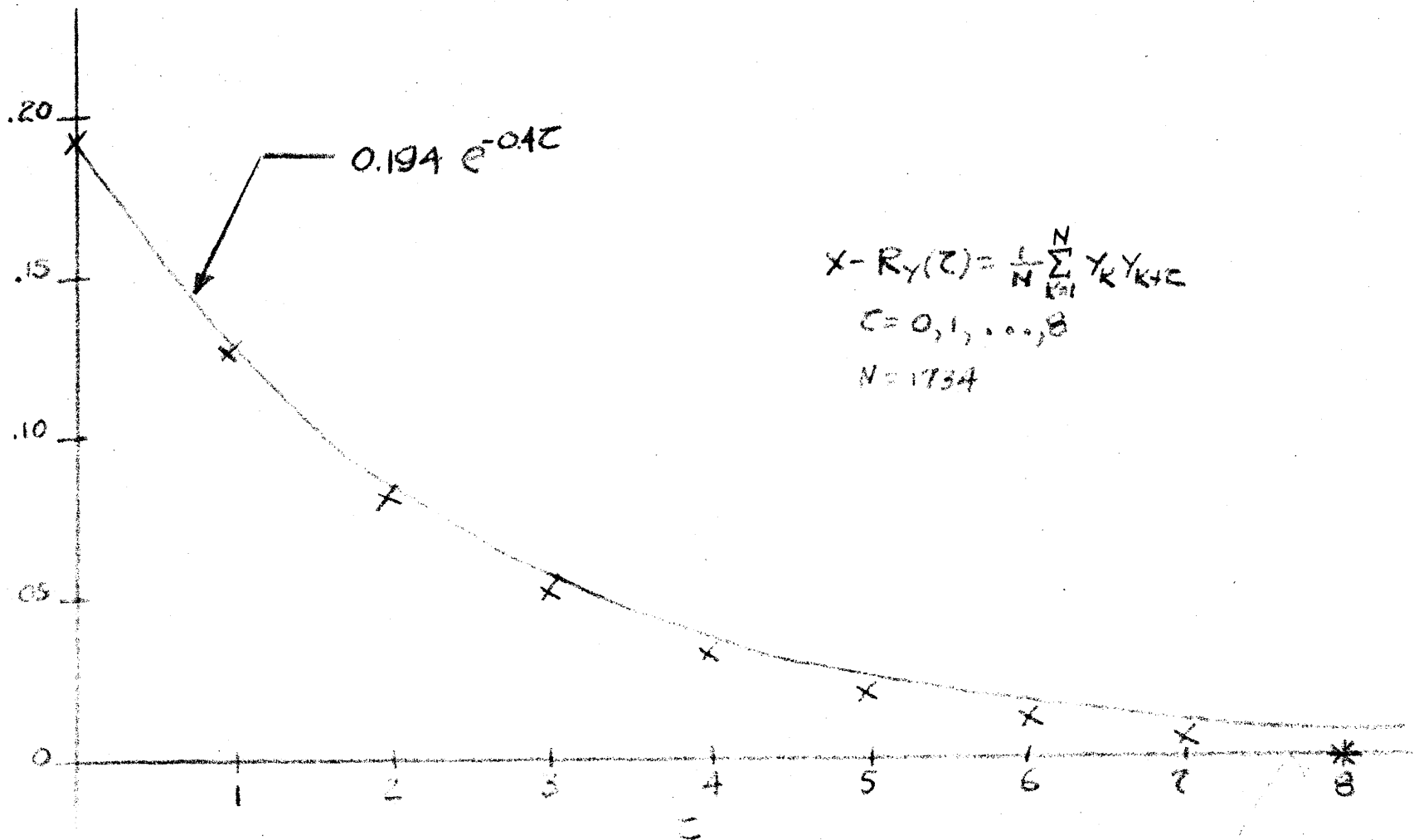Figure 6.   Output Distribution, Filtered Gaussian Random Numbers.

$$X - R_Y(\tau) = \frac{1}{N}\sum_{k=1}^{N} Y_k Y_{k+\tau}$$

$$\tau = 0, 1, \ldots, 8$$

$$N = 1734$$

$$0.194 \, e^{-0.4\tau}$$

Figure 7.   Output Autocorrelation Function, Filtered Gaussian Random Numbers.

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Appendix A   "Approximate - Gaussian" Random Number
                              Generator Subroutine

Enter Subroutine with + TRA 0006.0
Exit to next location with gaussian random number in A and R registers

0006.0	+	SAX	7760.0	+	CTL	0010.0
	+	CTV	0020.0	+	TRA	7760.0
0010.0	+	ADD	7773.0	+	STA	7773.1
	+	CLA	7777.0	+	MPY	7776.0
	+	XAR	0000.0	+	STO	7777.0
	+	ARS	0001.0	+	ADD	7774.0
	+	STO	7774.0	+	CLA	7777.0
	+	MPY	7776.0	+	XAR	0000.0
	+	STO	7777.0	+	ARS	0001.0
	+	ADD	7774.0	+	STO	7774.0
0020.0	+	CLA	7777.0	+	MPY	7776.0
	+	CLA	7775.0	+	XAR	0000.0
	+	STO	0027.0	+	ARS	0001.0
	+	FAD	7774.0	+	TRA	0000.1

$$-3 \text{ at Binary scale of } 2$$
$$+2 \text{ at Binary scale of } 39$$
$$N \text{ at Binary scale of } 39$$
$$x_0 = +1 \text{ at Binary scale of } 39$$

Notes:

(1) It is recommended that N be an odd power of 3 or 5.

(2) Main memory addresses are underlined.

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Appendix B   Odd Powers of 3 and 5

RECOMP users may find the following list of odd powers of 3 and 5, <u>in</u>
<u>Command</u> <u>format</u>, useful in programming random number generators:

3^{23}	=	+	1275321	+	6720451
3^{21}	=	+	0115731	+	4245311
3^{19}	=	+	0010520	-	6547551
3^{17}	=	+	0000751	-	2413411
3^{15}	=	+	0000061	+	2744651
5^{15}	=	+	0343271	+	5223061
5^{13}	=	+	0011060	-	2347121
5^{11}	=	+	0000270	+	1035561
5^{9}	=	+	0000001	+	5632621

- -

Appendix C   Probability Density Function and Cumulative

Distribution of $p_3(u)$ and $p_4(u)$

u	$p_3(u)$	$\int_0^u p_3(x)dx$	$p_4(u)$	$\int_0^u p_4(x)dx$
.0	+.37500 + 0	+.00000 + 0	+.66667 + 0	+.00000 + 0
.1	+.37375 + 0	+.37458 - 1	+.66343 + 0	+.38427 - 1
.2	+.37000 + 0	+.74667 - 1	+.65410 + 0	+.76489 - 1
.3	+.36375 + 0	+.11137 + 0	+.63926 + 0	+.11385 + 0
.4	+.35500 + 0	+.14733 + 0	+.61949 + 0	+.15021 + 0
.5	+.34375 + 0	+.18229 + 0	+.59536 + 0	+.18530 + 0
.6	+.33000 + 0	+.21600 + 0	+.56745 + 0	+.21888 + 0
.7	+.31375 + 0	+.24821 + 0	+.53634 + 0	+.25076 + 0
.8	+.29500 + 0	+.27867 + 0	+.50260 + 0	+.28076 + 0
.9	+.27375 + 0	+.30712 + 0	+.46681 + 0	+.30876 + 0
1.0	+.25000 + 0	+.33333 + 0	+.42956 + 0	+.33464 + 0
1.1	+.22563 + 0	+.35710 + 0	+.39141 + 0	+.35834 + 0
1.2	+.20250 + 0	+.37850 + 0	+.35294 + 0	+.37983 + 0
1.3	+.18063 + 0	+.39765 + 0	+.31474 + 0	+.39910 + 0
1.4	+.16000 + 0	+.41467 + 0	+.27737 + 0	+.41619 + 0
1.5	+.14063 + 0	+.42969 + 0	+.24143 + 0	+.43116 + 0
1.6	+.12250 + 0	+.44283 + 0	+.20747 + 0	+.44410 + 0
1.7	+.10563 + 0	+.45423 + 0	+.17609 + 0	+.45516 + 0
1.8	+.90000 - 1	+.46400 + 0	+.14781 + 0	+.46450 + 0
1.9	+.75625 - 1	+.47227 + 0	+.12273 + 0	+.47229 + 0
2.0	+.62500 - 1	+.47917 + 0	+.10067 + 0	+.47873 + 0
2.1	+.50625 - 1	+.48481 + 0	+.81415 - 1	+.48397 + 0
2.2	+.40000 - 1	+.48933 + 0	+.64791 - 1	+.48818 + 0
2.3	+.30625 - 1	+.49285 + 0	+.50599 - 1	+.49150 + 0
2.4	+.22500 - 1	+.49550 + 0	+.38647 - 1	+.49406 + 0
2.5	+.15625 - 1	+.49740 + 0	+.28743 - 1	+.49600 + 0
2.6	+.10000 - 1	+.49867 + 0	+.20695 - 1	+.49742 + 0
2.7	+.56250 - 2	+.49944 + 0	+.14309 - 1	+.49842 + 0
2.8	+.25000 - 2	+.49983 + 0	+.93944 - 2	+.49910 + 0
2.9	+.62500 - 3	+.49998 + 0	+.57576 - 2	+.49953 + 0
3.0	+.84703 -21	+.50000 + 0	+.32063 - 2	+.49979 + 0
3.1	+.00000 + 0	+.10000 + 1	+.15482 - 2	+.49992 + 0
3.2	+.00000 + 0	+.10000 + 1	+.59085 - 3	+.49998 + 0
3.3	+.00000 + 0	+.10000 + 1	+.14174 - 3	+.50000 + 0
3.4	+.00000 + 0	+.10000 + 1	+.84484 - 5	+.50000 + 0
3.5	+.00000 + 0	+.10000 + 1	+.00000 + 0	+.50000 + 0
3.6	+.00000 + 0	+.10000 + 1	+.00000 + 0	+.50000 + 0

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REFERENCES

a) "Quarterly of Applied Mathematics" Vol XVI, 1958